April 30, 1885.

THE PRESIDENT in the Chair.

The Presents received were laid on the table, and thanks ordered for them.

The following Papers were read:—

I. "Abstract of some Results in Elliptic Functions. (Part II.)" By JOHN GRIFFITHS, M.A. Communicated by Professor G. G. STOKES, Sec. R.S. Received April 9, 1885.

1. *On the $z(u)$ Function Complementary to Jacobi's* $\mathrm{Z}(u)$.

The double periodicity of the elliptic functions gives rise to an interesting function of the form $z(u)=a-\frac{\mathrm{J}'}{\mathrm{K}'}u$, where

$$u=\int_0^\theta \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}, \quad a=\int_0^\theta \sqrt{1-k^2\sin^2\theta}\, d\theta=\mathrm{E}(u), \ \mathrm{J}'=\mathrm{K}'-\mathrm{E}'.$$

By changing a, u, respectively, into (1) $a+2i\mathrm{J}'$, $u+2i\mathrm{K}'$, (2) $a+2\mathrm{E}$, $u+2\mathrm{K}$, (3) $a+\mathrm{E}-k^2\frac{\mathrm{sn}u\mathrm{cn}u}{\mathrm{dn}u}$, $u+\mathrm{K}$, it is easily seen that $z(u)$ satisfies the following relations, viz.:—

$$z(u+2i\mathrm{K}')=z(u),$$

$$z(u+2\mathrm{K})-z(u)=\frac{\pi}{\mathrm{K}'} \ \left(\text{since } \mathrm{KE}'+\mathrm{K}'\mathrm{E}-\mathrm{KK}'=\frac{\pi}{2}\right),$$

$$z(u+\mathrm{K})-z(u)=\frac{\pi}{2\mathrm{K}'}-k^2\frac{\mathrm{sn}u\mathrm{cn}u}{\mathrm{dn}u}.$$

2. *Deduction of a $\Phi(u)$ Function from $z(u)$.*

Writing $\Phi(u)=\sqrt{\frac{2k'\mathrm{K}'}{\pi}}\, e^{\int_0 z(u)du}$ we can take the foregoing $z(u)$ relations as equivalent to—

$$\Phi(u+2i\mathrm{K}')=-\Phi(u),$$

$$\Phi(u+2\mathrm{K})=\frac{1}{r}e^{\frac{\pi u}{\mathrm{K}}}\Phi(u),$$

$$\text{dn}u = \sqrt{k'}r^{\frac{1}{4}}e^{-\frac{\pi u}{2K'}}\frac{\Phi(u+K)}{\Phi(u)},$$

where $$r = e^{-\frac{\pi K}{K'}}$$

$\Phi(u)$ is, in fact, connected with Jacobi's $\Theta(u)$ by the equation $\Phi(u) \div \Phi(0) = e^{-\frac{\pi u^2}{4KK'}}\Theta(u) \div \Theta(0)$.

3. *Expansion of $\Phi(u)$ in a Hyper-harmonic Series containing odd Multiples of $\frac{\pi u}{2K'}$.*

From the above materials it is found that—

$$\Phi(u) = 2\left\{\sqrt[4]{r}\cosh\frac{\pi u}{2K'} + \sqrt[4]{r^9}\cosh\frac{3\pi u}{2K'} + \sqrt[4]{r^{25}}\cosh\frac{5\pi u}{2K'} + \ldots \textit{ad infin.}\right\},$$

where $$r = e^{-\frac{\pi K}{K'}} \text{ and } \cosh x = \tfrac{1}{2}(e^x + e^{-x}).$$

4. *Some Consequences of the above Theorems.*

Among the numerous results which flow from the above I notice the following, viz.:—

$$(\alpha.)\quad a - \frac{J'}{K'}u = \frac{\pi}{2K'}\frac{\sqrt[4]{r}\sinh\frac{\pi u}{2K'} + 3\sqrt[4]{r^9}\sinh\frac{3\pi u}{2K'} + \ldots}{\sqrt[4]{r}\cosh\frac{\pi n}{2K'} + \sqrt[4]{r^9}\cosh\frac{3\pi u}{2K'} + \ldots}.$$

If this be combined with Jacobi's

$$Z(u) = a - \frac{E}{K}u = \frac{2\pi}{K}\left\{\frac{q}{1-q^2}\sin\frac{\pi u}{K} + \ldots\right\}$$

we have the curious relation—

$$\frac{u}{{}^*KK'} = \frac{1}{K'}\frac{\sqrt[4]{r}\sinh\frac{\pi u}{2K'} + 3\sqrt[4]{r^9}\sinh\frac{3\pi u}{2K'} + 5\sqrt[4]{r^{25}}\sinh\frac{5\pi u}{2K'} + \ldots}{\sqrt[4]{r}\cosh\frac{\pi u}{2K'} + \sqrt[4]{r^9}\cosh\frac{3\pi u}{2K'} + \sqrt[4]{r^{25}}\cosh\frac{5\pi n}{2K'} + \ldots}$$

* Other relations follow from the z function $z(u) = a - \frac{E+iJ'}{K+iK'}u$, which deserves to be studied. As regards the *transformation* of the function Φ, the results are very similar to those obtained in the case of Jacobi's Θ.—April 29, 1885.

$$-\frac{4}{K}\left\{\frac{q}{1-q^2}\sin\frac{\pi u}{K}+\frac{q^2}{1-q^4}\sin\frac{2\pi u}{K}+\frac{q^3}{1-q^6}\sin\frac{3\pi u}{K}+\ \ldots\right\}.$$

(β.) Putting $u=2n\mathrm{K}$, we deduce a simple identity, viz.:—
If n be an integer, then—

$$(1+2n)r^n+(3+2n)r^{2+3n}+(5+2n)r^{6+5n}+(7+2n)r^{12+7n}+\ \ldots\ \textit{ad infin.}$$
$$=(1-2n)r^{-n}+(3-2n)r^{2-3n}+(5-2n)r^{6-5n}+(7-2n)r^{12-7n}+\ \ldots\ \textit{ad infin.}$$

(γ.) From the formula $\operatorname{dn}u=\sqrt{k'}r^{\frac{1}{4}}e^{-\frac{\pi u}{2K'}}\dfrac{\Phi(u+K)}{\Phi(u)}$,

we have
$$\sqrt{k'}=\frac{\Phi(0)}{r^{\frac{1}{4}}\Phi(K)}$$
$$=\frac{\{2\sqrt[4]{r}+\sqrt[4]{r^9}+\sqrt[4]{r^{25}}+\sqrt[4]{r^{49}}+\ \ldots\}}{r^{\frac{1}{4}}(r^{\frac{1}{2}}+r^{-\frac{1}{2}})+r^{\frac{5}{4}}(r^{\frac{3}{2}}+r^{-\frac{3}{2}})+r^{\frac{13}{4}}(r^{\frac{5}{2}}+r^{-\frac{5}{2}})+\ \ldots}$$
$$=2\,\frac{\sqrt[4]{r}+\sqrt[4]{r^9}+\sqrt[4]{r^{25}}+\sqrt{r^{49}}+\ \ldots}{1+2r+2r^4+2r^9+2r^{16}+\ \ldots}$$

This result is, in fact, Jacobi's

$$\sqrt{k}=2\,\frac{\sqrt[4]{q}+\sqrt[4]{q^9}+\sqrt[4]{q^{25}}+\ \ldots}{1+2q+2q^4+\ \ldots},$$ as we can see by changing k'

into k, and consequently r into q.

(δ.) From $\Phi(u)=\sqrt{\dfrac{2k'K'}{\pi}}e^{\int_0 z(u)\,du}$ we deduce

$$\sqrt{\frac{2k'K'}{\pi}}=\Phi(0)$$
$$=2\{\sqrt[4]{r}+\sqrt[4]{r^9}+\sqrt[4]{r^{25}}+\sqrt[4]{r^{49}}+\ \ldots\}$$

i.e., $$\sqrt{\frac{2K'}{\pi}}=1+2r+2r^4+2r^9+2r^{16}+\ \ldots$$

5. *Extension of the above Method to a $\zeta_1(u)$ Function connected with Elliptic Integrals of the Third Kind.*

In a former note by the present writer mention was made of a $\zeta(u)$ function of the form $\zeta(u)=\dfrac{\pi}{2(1-\mu)\Pi K'}\left(p-\dfrac{\Pi}{K}u\right)$,

where

$$p=\int_0^\theta\frac{d\theta}{(1+n\sin^2\theta)\sqrt{1-k^2\sin^2\theta}},\quad \Pi=\int_0^{\frac{\pi}{2}}\frac{d\theta}{(1+n\sin^2\theta)\sqrt{1-k^2\sin^2\theta}},$$

$$\mu=\frac{P'}{\Pi}\div\frac{K'}{K},\quad P'=K'-\frac{n}{1+n}\Pi',$$

$$\Pi' = \int_0^{\frac{\pi}{2}} \frac{d\theta}{(1+n'\sin^2\theta)\sqrt{1-k'^2\sin^2\theta}}, \quad n'(1+n) = -k'^2.$$

This is not exactly the form considered by Jacobi, but if we write $\frac{\Pi}{K} - 1 = \frac{\text{tn}u_0}{\text{dn}u_0}Z(u_0)$ and $n = -k^2 sn^2 u_0$ his result is equivalent to

$$\zeta(u) = \frac{1}{2u_0}\log\frac{\Theta(u+u_0)}{\Theta(u-u_0)}.$$

Connected with $\zeta(u)$ is a *second* function of the form

$$\zeta_1(u) = \frac{\pi}{2(1-\mu)\Pi K'}\left(p - \frac{P'}{K'}u\right).$$

This satisfies the relations

$$\left.\begin{array}{r}\zeta_1(u+2iK') = \zeta_1(u) \\ \zeta_1(u+2K) - \zeta_1(u) = \frac{\pi}{K'}\end{array}\right\},$$

and I find that it can be expressed in terms of $\Phi(u)$ by means of the equation $\zeta_1(u) = \frac{1}{2u_0}\log\frac{\Phi(u+u_0)}{\Phi(u-u_0)}$, where u_0 is the same constant as above.

It thus appears that $\Theta(u)$ and $\Phi(u)$ are connected with and supplement each other in a very remarkable manner.

For example, if we write $\zeta(u)$ and $\zeta_1(u)$ in the more convenient forms $\zeta(u, u_0)$, $\zeta_1(u, u_0)$, it follows that besides Jacobi's result, $u_0\zeta(u, u_0) = u\zeta(u_0, u)$, we have likewise the equivalent form $u_0\zeta_1(u, u_0) = u\zeta_1(u_0, u)$.

II. "Further Observations on Enterochlorophyll and Allied Pigments." By C. A. MacMunn, M.A., M.D. Communicated by Professor M. Foster, Sec. R.S. Received April 21, 1885.

(Abstract.)

In a paper read before the Royal Society in 1883, I described the results of an examination of the so-called "bile" of invertebrates, and showed that the alcohol extracts of their liver or other appendage of the intestine answering to that organ, showed a spectrum so like that of vegetable chlorophyll, as to have led me to assume that no essential difference exists between the spectrum of enterochlorophyll and plant chlorophyll.